One and Two-parameters Lindley Distributions: Ordinary Differential Equations

Hilary I. Okagbue, *Member, IAENG*, Pelumi E. Oguntunde, Abiodun A. Opanuga and Sheila A. Bishop

***Abstract*— Differential calculus was applied to get the ordinary differential equations (ODE) of the probability functions of the one and two-parameters Lindley distributions. The parameters and support that characterized the distribution inevitably determine the nature, existence, uniqueness and solution of the ODEs. The method is recommended to be applied to other probability distributions and probability functions not considered in this paper. Computer codes and programs can be used for the implementation.**

***Index Terms*— Differential calculus, quantile function, hazard function, reversed hazard function, inverse survival function, probability density function, Lindley distribution.**

I. Introduction

CALCULUS in general and differential calculus in particular is often used in statistics in parameter and modal estimations. The method of maximum likelihood is an example.

Differential equations often arise from the understanding and modeling of real life problems or some observed physical phenomena. Approximations of probability functions are one of the major areas of application of calculus and ordinary differential equations in mathematical statistics. The approximations are helpful in the recovery of the probability functions of complex distributions [1-4].

Apart from mode estimation, parameter estimation and approximation, probability density function (PDF) of distributions can be transformed as ODE whose solution yields the respective PDF. Some of which are available. See [5-9] for details.

The aim of this research is to obtain homogenous ordinary differential equations for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the one-parameter and two-parameter Lindley distributions. This will also help to provide the answers as to whether there are discrepancies between the support of the distributions and the conditions necessary for the behavior and existence of the ODEs. Similar results for other distributions have been proposed, see [10-23] for details.

Manuscript received February 9, 2018; revised March 14, 2018. This work was sponsored by Covenant University, Ota, Nigeria.

H. I. Okagbue, P. E. Oguntunde, A. A. Opanuga and S. A. Bishop are with the Department of Mathematics, Covenant University, Ota, Nigeria.
hilary.okagbue@covenantuniversity.edu.ng
pelumi.oguntunde@covenantuniversity.edu.ng
abiodun.opanuga @covenantuniversity.edu.ng
sheila.bishop@covenantuniversity.edu.ng

One parameter and two-parameter Lindley distributions were considered in this research. The details of the distribution can be obtained in [24-26].
Krishna and Kumar [27] studied the effects of censored samples in the reliability analysis of the distribution while Gupta and Singh [28] looked at the parameters estimation of the distribution with hybrid censored data.

Some of the generalizations or modification of the distribution include: the discrete Poisson-Lindley distribution [29], 3-parameter generalization of the 1-parameter Lindley distribution [30], size-biased Poisson-Lindley distribution [31], negative binomial-Lindley distribution [32], discrete Lindley distribution [33], two-parameter weighted Lindley distribution [34], an extended Lindley distribution [35], Power Lindley distribution [36] and generalized Lindley distribution [37].

Also available are: transmuted Lindley distribution [38], two parameter quasi Lindley distribution [39], Kumaraswamy Quasi Lindley [40], transmuted Quasi Lindley distribution [41], beta-Lindley distribution [42], inverse Lindley distribution [43], truncated Lindley distribution [44], Pareto Poisson–Lindley distribution [45], Log–Lindley distribution [46], generalized Power Lindley distribution [47], a new generalization of the distribution based on the probabilistic mixture of two gamma distributions [48] and Lindley exponential distribution [49-50].

Also available are: gamma Lindley distribution [51], Lindley-Poisson distribution [52], Marshall-Olkin extended Lindley distribution [53], a new 4-parameter beta Lindley distribution [54], a new three parameter Lindley distribution [55], generalized inverse Lindley distribution [56], extended inverse Lindley distribution [57], truncated Lindley gamma distribution [58], two parameter discrete Lindley distribution [59], five parameter Lindley distribution [60], Lindley slash distribution [61], wrapped Lindley distribution [62], transmuted two-parameter Lindley distribution [63] and multivariate Lindley distribution [64].

Modeling risk of lifetime data by [65] is one of the various applications of the distribution available.

Differential calculus was used to obtain the results.

II. One-Parameter Lindley Distribution

A. Probability Density Function

The PDF of the one-parameter Lindley distribution is given as;

$$f(x)=\frac{\theta^2}{\theta+1}(x+1)\mathrm{e}^{-\theta x} \tag{1}$$

Differentiate equation (1);

$$f'(x)=\left\{\frac{1}{x+1}-\frac{\theta\mathrm{e}^{-\theta x}}{\mathrm{e}^{-\theta x}}\right\}f(x) \tag{2}$$

The equation can only exists for $x,\theta>0$.

$$f'(x)=\left\{\frac{1}{x+1}-\theta\right\}f(x) \tag{3}$$

The first order ODE for the PDF of the one-parameter Lindley distribution is given as;

$$(x+1)f'(x)+(\theta(x+1)-1)f(x)=0 \tag{4}$$

$$f(1)=\frac{2\theta^2\mathrm{e}^{-\theta}}{\theta+1} \tag{5}$$

See [10-23] for details.

B. Quantile Function

The QF of the one-parameter Lindley distribution can be obtained from its cumulative distribution function (CDF);

$$F(t)=1-\frac{(\theta t+\theta+1)\mathrm{e}^{-\theta t}}{\theta+1} \tag{6}$$

$$p=1-\frac{(\theta Q(p)+\theta+1)\mathrm{e}^{-\theta Q(p)}}{\theta+1} \tag{7}$$

$$\frac{(\theta Q(p)+\theta+1)\mathrm{e}^{-\theta Q(p)}}{\theta+1}=1-p \tag{8}$$

$$(\theta Q(p)+\theta+1)\mathrm{e}^{-\theta Q(p)}=(\theta+1)(1-p) \tag{9}$$

Taking logarithmic on both sides;

$$-\theta Q(p)+\ln(\theta Q(p)+\theta+1)=\ln(\theta+1)+\ln(1-p) \tag{10}$$

Differentiate equation (10);

$$-\theta Q'(p)+\frac{\theta Q'(p)}{(\theta Q(p)+\theta+1)}=-\frac{1}{1-p} \tag{11}$$

The equation can only exists for $\theta>0, 0<p<1$.

$$\theta Q'(p)-\frac{\theta Q'(p)}{(\theta Q(p)+\theta+1)}=\frac{1}{1-p} \tag{12}$$

$$\frac{(\theta Q(p)+\theta+1)\theta Q'(p)-\theta Q'(p)}{\theta Q(p)+\theta+1}=\frac{1}{1-p} \tag{13}$$

$$(\theta Q(p)+\theta+1-1)\theta Q'(p)=\frac{\theta Q(p)+\theta+1}{1-p} \tag{14}$$

$$(Q(p)+1)\theta^2 Q'(p)=\frac{\theta Q(p)+\theta+1}{1-p} \tag{15}$$

$$(1-p)(Q(p)+1)\theta^2 Q'(p)=\theta Q(p)+\theta+1 \tag{16}$$

The first order ODE for the QF of the one-parameter Lindley distribution is given as;

$$(1-p)(Q(p)+1)\theta^2 Q'(p)-\theta Q(p)-\theta-1=0 \tag{17}$$

$$\ln(\theta Q(0.1)+\theta+1)-\theta Q(0.1)=\ln(\theta+1)+\ln 0.9 \tag{18}$$

See [10-23] for details.

C. Survival Function

The SF of the one-parameter Lindley distribution is given by;

$$S(t)=\frac{(\theta t+\theta+1)\mathrm{e}^{-\theta t}}{\theta+1} \tag{19}$$

Differentiate equation (19), to obtain the first order ODE;

$$S'(t)=-\frac{\theta^2}{\theta+1}(t+1)\mathrm{e}^{-\theta t} \tag{20}$$

The equation can only exists for $t,\theta>0$.

The alternative to equation (19) is given as;

$$\mathrm{e}^{-\theta t}=\frac{(\theta+1)S(t)}{\theta t+\theta+1} \tag{21}$$

Substitute equation (21) into equation (20);

$$S'(t)=-\frac{\theta^2(t+1)S(t)}{\theta t+\theta+1} \tag{22}$$

The first order ODE for the SF of the one-parameter Lindley distribution is given as;

$$(\theta t+\theta+1)S'(t)+\theta^2(t+1)S(t)=0 \tag{23}$$

$$S(1)=\frac{(2\theta+1)\mathrm{e}^{-\theta}}{\theta+1} \tag{24}$$

See [10-23] for details.

D. Inverse Survival Function

The ISF of the one-parameter Lindley distribution can be obtained from the survival function;

$$p=\frac{(\theta Q(p)+\theta+1)\mathrm{e}^{-\theta Q(p)}}{\theta+1} \tag{25}$$

$$(\theta Q(p)+\theta+1)\mathrm{e}^{-\theta Q(p)}=(\theta+1)p \tag{26}$$

Taking logarithmic on both sides;

$$-\theta Q(p)+\ln(\theta Q(p)+\theta+1)=\ln(\theta+1)+\ln p \tag{27}$$

Differentiate equation (27);

$$-\theta Q'(p)+\frac{\theta Q'(p)}{(\theta Q(p)+\theta+1)}=\frac{1}{p} \tag{28}$$

$$\theta Q'(p)-\frac{\theta Q'(p)}{(\theta Q(p)+\theta+1)}=-\frac{1}{p} \tag{29}$$

The equation can only exists for $\theta>0, 0<p<1$.

$$\frac{(\theta Q(p)+\theta+1)\theta Q'(p)-\theta Q'(p)}{\theta Q(p)+\theta+1}=-\frac{1}{p} \tag{30}$$

$$(\theta Q(p)+\theta+1-1)\theta Q'(p)=-\frac{(\theta Q(p)+\theta+1)}{p} \tag{31}$$

ISBN: 978-988-14048-1-7
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$(Q(p)+1)\theta^2 Q'(p) = -\frac{(\theta Q(p)+\theta+1)}{p} \quad (32)$$

$$\theta^2 p(Q(p)+1)Q'(p) = -(\theta Q(p)+\theta+1) \quad (33)$$

The first order ODE for the ISF of the one-parameter Lindley distribution is given as;

$$\theta^2 p(Q(p)+1)Q'(p)+\theta Q(p)+\theta+1=0 \quad (34)$$

$$\ln(\theta Q(0.1)+\theta+1)-\theta Q(0.1) = \ln(\theta+1)+\ln 0.1 \quad (35)$$

See [10-23] for details.

E. Hazard Function

The HF of the one-parameter Lindley distribution is given by;

$$h(t) = \frac{\theta^2(t+1)}{\theta t+\theta+1} \quad (36)$$

Differentiate equation (36), to obtain the first order ODE;

$$h'(t) = \left\{\frac{1}{t+1} - \frac{\theta(\theta t+\theta+1)^{-2}}{(\theta t+\theta+1)^{-1}}\right\} h(t) \quad (37)$$

$$h'(t) = \left\{\frac{1}{t+1} - \frac{\theta}{(\theta t+\theta+1)}\right\} h(t) \quad (38)$$

The equation can only exists for $t, \theta > 0.$

Equation (36) can also be written as;

$$\frac{h(t)}{\theta(t+1)} = \frac{\theta}{\theta t+\theta+1} \quad (39)$$

Substitute equation (39) into equation (38);

$$h'(t) = \left\{\frac{1}{t+1} - \frac{h(t)}{\theta(t+1)}\right\} h(t) \quad (40)$$

The first order ODE for the HFof the one-parameter Lindley distribution is given as;

$$(t+1)h'(t)+h^2(t)-\theta h(t) = 0 \quad (41)$$

$$h(1) = \frac{2\theta^2}{2\theta+1} \quad (42)$$

F. Reversed Hazard Function

The RHF of the one-parameter Lindley distribution is given by;

$$j(t) = \frac{\theta^2(t+1)\,\mathrm{e}^{-\theta t}}{(\theta+1)-(\theta t+\theta+1)\,\mathrm{e}^{-\theta t}} \quad (43)$$

Differentiate equation (43);

$$j'(t) = \left\{\begin{array}{l} \frac{1}{t+1} - \frac{\theta\,\mathrm{e}^{-\theta t}}{\mathrm{e}^{-\theta t}} - \\ \frac{\theta^2(t+1)\,\mathrm{e}^{-\theta t}((\theta+1)-(\theta t+\theta+1)\,\mathrm{e}^{-\theta t})^{-2}}{((\theta+1)-(\theta t+\theta+1)\,\mathrm{e}^{-\theta t})^{-1}} \end{array}\right\} j(t) \quad (44)$$

$$j'(t) = \left\{\frac{1}{t+1} - \theta - \frac{\theta^2(t+1)\,\mathrm{e}^{-\theta t}}{(\theta+1)-(\theta t+\theta+1)\,\mathrm{e}^{-\theta t}}\right\} j(t) \quad (45)$$

The equation can only exists for $t, \theta > 0.$

$$j'(t) = \left\{\frac{1}{t+1} - \theta - j(t)\right\} j(t) \quad (46)$$

The first order ODE for the RHF of the one-parameter Lindley distribution is given as;

$$(t+1)\,j'(t)-(1-(t+1)\theta)\,j(t)+(t+1)\,j^2(t) = 0 \quad (47)$$

$$j(1) = \frac{2\theta^2\,\mathrm{e}^{-\theta}}{(\theta+1)-(2\theta+1)\,\mathrm{e}^{-\theta}} = \frac{2\theta^2}{(\theta+1)\,\mathrm{e}^{\theta}-(2\theta+1)} \quad (48)$$

III. Two-Parameter Lindley Distribution

A. Probability Density Function

The PDF of the two- parameter Lindley distribution is given as;

$$f(x) = \frac{\theta^2}{\theta+\beta}(\beta x+1)\,\mathrm{e}^{-\theta x} \quad (49)$$

Differentiate equation (49);

$$f'(x) = \left\{\frac{\beta}{\beta x+1} - \frac{\theta\,\mathrm{e}^{-\theta x}}{\mathrm{e}^{-\theta x}}\right\} f(x) \quad (50)$$

$$f'(x) = \left\{\frac{\beta}{\beta x+1} - \theta\right\} f(x) \quad (51)$$

The equation can only exists for $x, \beta, \theta > 0.$

The first order ODE for the PDF of the two-parameter Lindley distribution is given as;

$$(\beta x+1)\,f'(x)-(\beta-\theta(\beta x+1))\,f(x) = 0 \quad (52)$$

$$f(1) = \frac{\theta^2(1+\beta)\,\mathrm{e}^{-\theta}}{\theta+\beta} \quad (53)$$

B. Quantile Function

The QF of the two-parameter Lindley distribution can be obtained from the cumulative distribution function;

$$F(t) = 1 - \frac{(\theta\beta t+\theta+\beta)\,\mathrm{e}^{-\theta t}}{\theta+\beta} \quad (54)$$

$$p = 1 - \frac{(\theta\beta Q(p)+\theta+\beta)\,\mathrm{e}^{-\theta Q(p)}}{\theta+\beta} \quad (55)$$

$$\frac{(\beta\theta Q(p)+\theta+\beta)\,\mathrm{e}^{-\theta Q(p)}}{\theta+\beta} = 1-p \quad (56)$$

$$(\beta\theta Q(p)+\theta+\beta)\,\mathrm{e}^{-\theta Q(p)} = (\theta+\beta)(1-p) \quad (57)$$

ISBN: 978-988-14048-1-7
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Taking logarithmic on both sides;

$$-\theta Q(p)+\ln(\beta\theta Q(p)+\theta+\beta)=\ln(\theta+\beta)+\ln(1-p) \quad (58)$$

Differentiate equation (58);

$$-\theta Q'(p)+\frac{\beta\theta Q'(p)}{\beta\theta Q(p)+\theta+\beta}=-\frac{1}{1-p} \quad (59)$$

$$\theta Q'(p)-\frac{\beta\theta Q'(p)}{\beta\theta Q(p)+\theta+\beta}=\frac{1}{1-p} \quad (60)$$

$$\frac{(\beta\theta Q(p)+\theta+\beta)\theta Q'(p)-\beta\theta Q'(p)}{\theta Q(p)+\theta+\beta}=\frac{1}{1-p} \quad (61)$$

The equation can only exists for $\theta,\beta>0, 0<p<1.$

$$(\beta\theta Q(p)+\theta+\beta-\beta)\theta Q'(p)=\frac{\beta\theta Q(p)+\theta+\beta}{1-p} \quad (62)$$

$$(\beta Q(p)+1)\theta^2 Q'(p)=\frac{\beta\theta Q(p)+\theta+\beta}{1-p} \quad (63)$$

$$(1-p)(\beta Q(p)+1)\theta^2 Q'(p)=\beta\theta Q(p)+\theta+\beta \quad (64)$$

The first order ODE for the QF of the two-parameter Lindley distribution is given as;

$$(1-p)(\beta Q(p)+1)\theta^2 Q'(p)-\beta\theta Q(p)-\theta-\beta=0 \quad (65)$$

$$\ln(\beta\theta Q(0.1)+\theta+\beta)-\theta Q(0.1)=\ln(\theta+\beta)+\ln 0.9 \quad (66)$$

C. Survival Function

The SF of the two-parameter Lindley distribution is given by;

$$S(t)=\frac{(\beta\theta t+\theta+\beta)\,\mathrm{e}^{-\theta t}}{\theta+\beta} \quad (67)$$

Differentiate equation (67) to obtain the first order ODE;

$$S'(t)=-\frac{\theta^2}{\theta+\beta}(\beta t+1)\,\mathrm{e}^{-\theta t} \quad (68)$$

Equation (67) can also be written as;

$$\frac{S(t)}{\beta\theta t+\theta+\beta}=\frac{\mathrm{e}^{-\theta t}}{\theta+\beta} \quad (69)$$

The equation can only exists for $t,\beta,\theta>0.$

Substitute equation (69) into equation (68);

$$S'(t)=-\frac{\theta^2(\beta t+1)S(t)}{\beta\theta t+\theta+\beta} \quad (70)$$

The first order ODE for the SF of the two-parameter Lindley distribution is given as;

$$(\beta\theta t+\theta+\beta)S'(t)+\theta^2(\beta t+1)S(t)=0 \quad (71)$$

$$S(1)=\frac{(\theta+\beta+\beta\theta)\,\mathrm{e}^{-\theta}}{\theta+\beta} \quad (72)$$

D. Inverse Survival Function

The ISF of the two-parameter Lindley distribution can be obtained from the survival function;

$$p=\frac{(\beta\theta Q(p)+\theta+\beta)\,\mathrm{e}^{-\theta Q(p)}}{\theta+\beta} \quad (73)$$

$$(\beta\theta Q(p)+\theta+\beta)\,\mathrm{e}^{-\theta Q(p)}=(\theta+\beta)p \quad (74)$$

Taking logarithmic on both sides;

$$-\theta Q(p)+\ln(\beta\theta Q(p)+\theta+\beta)=\ln(\theta+\beta)+\ln p \quad (75)$$

Differentiate equation (75);

$$-\theta Q'(p)+\frac{\beta\theta Q'(p)}{(\beta\theta Q(p)+\theta+\beta)}=\frac{1}{p} \quad (76)$$

$$\theta Q'(p)-\frac{\beta\theta Q'(p)}{(\beta\theta Q(p)+\theta+\beta)}=-\frac{1}{p} \quad (77)$$

$$\frac{(\theta Q(p)+\theta+\beta)\theta Q'(p)-\beta\theta Q'(p)}{\theta Q(p)+\theta+\beta}=-\frac{1}{p} \quad (78)$$

The equation can only exists for $\theta,\beta>0, 0<p<1.$

$$(\beta\theta Q(p)+\theta+\beta-\beta)\theta Q'(p)=-\frac{(\beta\theta Q(p)+\theta+\beta)}{p} \quad (79)$$

$$(\beta Q(p)+1)\theta^2 Q'(p)=-\frac{(\beta\theta Q(p)+\theta+\beta)}{p} \quad (80)$$

$$\theta^2 p(\beta Q(p)+1)Q'(p)=-(\beta\theta Q(p)+\theta+\beta) \quad (81)$$

The first order ODE for the ISF of the two-parameter Lindley distribution is given as;

$$\theta^2 p(\beta Q(p)+1)Q'(p)+\beta\theta Q(p)+\theta+\beta=0 \quad (82)$$

$$\ln(\beta\theta Q(0.1)+\theta+\beta)-\theta Q(0.1)=\ln(\theta+\beta)+\ln 0.1 \quad (83)$$

E. Hazard Function

The HF of the two-parameter Lindley distribution is given by;

$$h(t)=\frac{\theta^2(\beta t+1)}{\beta\theta t+\theta+\beta} \quad (84)$$

Differentiate equation (84), to obtain the first order ODE;

$$h'(t)=\left\{\frac{\beta}{\beta t+1}-\frac{\theta\beta(\beta\theta t+\theta+\beta)^{-2}}{(\beta\theta t+\theta+\beta)^{-1}}\right\}h(t) \quad (85)$$

$$h'(t)=\left\{\frac{\beta}{\beta t+1}-\frac{\theta\beta}{(\beta\theta t+\theta+\beta)}\right\}h(t) \quad (86)$$

The condition necessary for the existence of the equation is $t,\beta,\theta>0.$

Alternatively, equation (84) is given as;

$$\frac{\beta h(t)}{\theta(\beta t+1)}=\frac{\theta\beta}{\beta\theta t+\theta+\beta} \quad (87)$$

Substitute equation (87) into equation (86);

$$h'(t)=\left\{\frac{\beta}{t+1}-\frac{\beta h(t)}{\theta(\beta t+1)}\right\}h(t) \qquad (88)$$

The first order ODE for the HF of the two-parameter Lindley distribution is given as;

$$(t+1)h'(t)+\beta h^2(t)-\beta\theta h(t)=0 \qquad (89)$$

$$h(1)=\frac{\theta^2(\beta+1)}{\beta\theta+\theta+\beta} \qquad (90)$$

F. Reversed Hazard Function

The RHF of the two-parameter Lindley distribution is given by;

$$j(t)=\frac{\theta^2(\beta t+1)\mathrm{e}^{-\theta t}}{(\theta+\beta)-(\beta\theta t+\theta+\beta)\mathrm{e}^{-\theta t}} \qquad (91)$$

Differentiate equation (91) to obtain the first order ODE;

$$j'(t)=\left\{\begin{array}{c}\dfrac{\beta}{\beta t+1}-\dfrac{\theta\,\mathrm{e}^{-\theta t}}{\mathrm{e}^{-\theta t}}\\ \theta^2(\beta t+1)\mathrm{e}^{-\theta t}((\theta+\beta)\\ -\dfrac{-(\beta\theta t+\theta+\beta)\mathrm{e}^{-\theta t})^{-2}}{((\theta+\beta)-(\beta\theta t+\theta+\beta)\mathrm{e}^{-\theta t})^{-1}}\end{array}\right\}j(t) \qquad (92)$$

$$j'(t)=\left\{\frac{\beta}{\beta t+1}-\theta-\frac{\theta^2(\beta t+1)\mathrm{e}^{-\theta t}}{(\theta+\beta)-(\beta\theta t+\theta+\beta)\mathrm{e}^{-\theta t}}\right\}j(t) \qquad (93)$$

The equation can only exists for $t,\beta,\theta>0$.

$$j'(t)=\left\{\frac{\beta}{\beta t+1}-\theta-j(t)\right\}j(t) \qquad (94)$$

The first order ODE for the RHF of the two-parameter Lindley distribution is given as;

$$(\beta t+1)j'(t)-(\beta-\theta(\beta t+1))j(t)+(\beta t+1)j^2(t)=0 \qquad (95)$$

$$j(1)=\frac{\theta^2(\beta+1)\mathrm{e}^{-\theta}}{(\theta+\beta)-(\theta\beta+\theta+\beta)\mathrm{e}^{-\theta}}=\frac{\theta^2(\beta+1)}{(\theta+\beta)\mathrm{e}^{\theta}-(\theta\beta+\theta+\beta)} \qquad (96)$$

IV. Concluding Remarks

Ordinary differential equations (ODEs) have been obtained for the probability functions of of one-parameter and two-parameter Lindley distributions. This differential calculus and efficient algebraic simplifications were used to derive the various classes of the ODEs. The parameter and the supports that characterize the distributions determine the nature, existence, orientation and uniqueness of the ODEs. The results are in agreement with those available in scientific literature. Furthermore several methods can be used to obtain desirable solutions to the ODEs [66-78]. This method of characterizing distributions cannot be applied to distributions whose PDF or CDF are either not differentiable or the domain of the support of the distribution contains singular points.

Acknowledgment

The comments of the reviewers were very helpful and led to an improvement of the paper. This research benefited from sponsorship from the Statistics sub-cluster of the *Industrial Mathematics Research Group* (TIMREG) of Covenant University and *Centre for Research, Innovation and Discovery* (CUCRID), Covenant University, Ota, Nigeria.

References

[1] J. Leydold and W. Hörmann, “ Generating generalized inverse Gaussian random variates by fast inversion,” *Comput. Stat. Data Analy.*, vol. 55, no. 1, pp. 213-217, 2011.

[2] G. Steinbrecher, G. and W.T. Shaw, “Quantile mechanics” *Euro. J. Appl. Math.*, vol. 19, no. 2, pp. 87-112, 2008.

[3] H.I. Okagbue, M.O. Adamu and T.A. Anake “Quantile Approximation of the Chi-square Distribution using the Quantile Mechanics,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 477-483.

[4] H.I. Okagbue, M.O. Adamu and T.A. Anake “Solutions of Chi-square Quantile Differential Equation,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 813-818.

[5] W.P. Elderton, Frequency curves and correlation, Charles and Edwin Layton. London, 1906.

[6] N. Balakrishnan and C.D. Lai, Continuous bivariate distributions, 2nd edition, Springer New York, London, 2009.

[7] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous Univariate Distributions, Volume 2. 2nd edition, Wiley, 1995.

[8] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous univariate distributions, Wiley New York. ISBN: 0-471-58495-9, 1994.

[9] H. Rinne, Location scale distributions, linear estimation and probability plotting using MATLAB, 2010.

[10] H.I. Okagbue, P.E. Oguntunde, A.A. Opanuga, E.A. Owoloko “Classes of Ordinary Differential Equations Obtained for the Probability Functions of Fréchet Distribution,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 186-191.

[11] H.I. Okagbue, P.E. Oguntunde, P.O. Ugwoke, A.A. Opanuga “Classes of Ordinary Differential Equations Obtained for the Probability Functions of Exponentiated Generalized Exponential Distribution,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 192-197.

[12] H.I. Okagbue, A.A. Opanuga, E.A. Owoloko, M.O. Adamu “Classes of Ordinary Differential Equations Obtained for the Probability Functions of Cauchy, Standard Cauchy and Log-Cauchy Distributions,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 198-204.

[13] H.I. Okagbue, S.A. Bishop, A.A. Opanuga, M.O. Adamu “Classes of Ordinary Differential Equations Obtained for the Probability Functions of Burr XII and Pareto Distributions,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 399-404.

[14] H.I. Okagbue, M.O. Adamu, E.A. Owoloko and A.A. Opanuga “Classes of Ordinary Differential Equations Obtained for the Probability Functions of Gompertz and Gamma Gompertz Distributions,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and

Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 405-411.

[15] H.I. Okagbue, M.O. Adamu, A.A. Opanuga and J.G. Oghonyon "Classes of Ordinary Differential Equations Obtained for the Probability Functions of 3-Parameter Weibull Distribution," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 539-545.

[16] H.I. Okagbue, A.A. Opanuga, E.A. Owoloko and M.O. Adamu "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Exponentiated Fréchet Distribution," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 546-551.

[17] H.I. Okagbue, M.O. Adamu, E.A. Owoloko and S.A. Bishop "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Half-Cauchy and Power Cauchy Distributions," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 552-558.

[18] H.I. Okagbue, P.E. Oguntunde, A.A. Opanuga and E.A. Owoloko "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Exponential and Truncated Exponential Distributions," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 858-864.

[19] H.I. Okagbue, O.O. Agboola, P.O. Ugwoke and A.A. Opanuga "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Exponentiated Pareto Distribution," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 865-870.

[20] H.I. Okagbue, O.O. Agboola, A.A. Opanuga and J.G. Oghonyon "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Gumbel Distribution," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 871-875.

[21] H.I. Okagbue, O.A. Odetunmibi, A.A. Opanuga and P.E. Oguntunde "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Half-Normal Distribution," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 876-882.

[22] H.I. Okagbue, M.O. Adamu, E.A. Owoloko and E.A. Suleiman "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Harris Extended Exponential Distribution," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 883-888.

[23] H.I. Okagbue, M.O. Adamu, T.A. Anake Ordinary Differential Equations of the Probability Functions of Weibull Distribution and their application in Ecology, *Int. J. Engine. Future Tech*., vol. 15, no. 4, pp. 57-78, 2018.

[24] D.V. Lindley, "Fiducial distributions and Bayes' theorem", *J. Roy. Soc. Series B*, vol. 20, pp. 102-107, 1958.

[25] M.E. Ghitany, B. Atieh and S. Nadarajah, "Lindley distribution and its application", *Math. Comp. Simul.*, vol. 78, no. 4, pp. 493-506, 2008.

[26] R. Shanker, S. Sharma and R. Shanker, "A two-parameter Lindley distribution for modeling waiting and survival times data", *Appl. Math.*, vol. 4, no. 2, pp. 363-368, 2013.

[27] H. Krishna and K. Kumar, "Reliability estimation in Lindley distribution with progressively type II right censored sample", *Math. Comp. Simul.*, vol. 82, no. 2, pp. 281-294, 2011.

[28] P.K. Gupta and B. Singh, "Parameter estimation of Lindley distribution with hybrid censored data", *Int. J. Syst. Assur. Engine. Magt.*, vol. 4, no. 4, pp. 378-385, 2013.

[29] M. Sankaran, "275. note: The discrete Poisson-Lindley distribution", *Biometrics*, pp. 145-149, 1970.

[30] H. Zakerzadeh, H., and A. Dolati, "Generalized Lindley Distribution", *J. Math. Exten*., vol. 3, no. 2, pp. 1-17, 2009.

[31] M.E. Ghitany and D.K. Al-Mutairi, "Size-biased Poisson-Lindley distribution and its application", *Metron*, vol. 66, no. 3, pp. 299-311, 2008.

[32] H. Zamani and N. Ismail, "Negative binomial-Lindley distribution and its application", *J. Math. Stat.*, vol. 6, no. 1, pp. 4-9, 2010.

[33] E. Gómez-Déniz and E. Calderín-Ojeda, "The discrete Lindley distribution: properties and applications", *J. Stat. Comput. Simul.*, vol. 81, no. 11, pp. 1405-1416, 2011.

[34] M.E. Ghitany, F. Alqallaf, D.K. Al-Mutairi and H.A. Husain, "A two-parameter weighted Lindley distribution and its applications to survival data", *Math. Comp. Simul.*, vol. 81, no. 6, pp. 1190-1201, 2011.

[35] H.S. Bakouch, B.M. Al-Zahrani, A.A. Al-Shomrani, V.A. Marchi and F. Louzada, "An extended Lindley distribution", *J. Korean Stat. Soc.*, vol. 41, no. 1, pp. 75-85, 2012.

[36] M.E. Ghitany, D.K. Al-Mutairi, N. Balakrishnan and L.J. Al-Enezi, "Power Lindley distribution and associated inference. *Comput. Stat. Data Analy.*, vol. 64, pp. 20-33, 2013.

[37] S. Nadarajah, H.S. Bakouch and R. Tahmasbi, "A generalized Lindley distribution", *Sankhya B*, vol. 73, no. 2, pp. 331-359, 2011.

[38] F. Merovci, "Transmuted Lindley distribution", *Inter. J. Open Probl. Comp. Sci. Math.*, vol. 6, no. 2, pp. 63-72, 2013.

[39] R. Shanker and A. Mishra, "A quasi Lindley distribution", *Afr. J. Math. Comp. Sci. Res.*, vol. 6, no. 4, pp. 64-71, 2013.

[40] I. Elbatal and M. Elgarhy, "Statistical properties of Kumaraswamy Quasi Lindley distribution", *Int. J. Math. Trends Tech.*, vol. 4, pp. 237-246, 2013.

[41] I. Elbatal and M. Elgarhy, "Transmuted Quasi Lindley distribution: A generalization of the Quasi Lindley distribution", *Int. J. Pure Appl. Sci. Tech.*, vol. 18, no. 2, pp. 59-70, 2013.

[42] F. Merovci and V.K. Sharma, "The beta-Lindley distribution: properties and applications", *J. Appl. Math.*, vol. 2014, Article ID 198951, 2014.

[43] V.K. Sharma, S.K. Singh, U. Singh and V. Agiwal, "The inverse Lindley distribution: a stress-strength reliability model with application to head and neck cancer data", *J. Ind. Prod. Engine.*, vol. 32, no. 3, pp. 162-173, 2015.

[44] S.K. Singh, U. Singh and V.K. Sharma, "The truncated Lindley distribution: Inference and application", *J. Stat. Appl. Prob.*, vol. 3, no. 2, pp. 219-228, 2014.

[45] A. Asgharzadeh, H.S. Bakouch and L. Esmaeili, "Pareto Poisson–Lindley distribution with applications:, *J. Appl. Stat.*, vol. 40, no. 8, pp. 1717-1734, 2013.

[46] E. Gómez-Déniz, M.A. Sordo and E. Calderín-Ojeda, "The Log–Lindley distribution as an alternative to the beta regression model with applications in insurance", *Insur. Math. Econ.*, vol. 54, 49-57, 2014.

[47] G. Warahena-Liyanage and M. Pararai, "A Generalized Power Lindley Distribution with Applications:, *Asian J. Math. Appl.*, Article ID: ama0169, 2014.

[48] A.M. Abouammoh, A.M. Alshangiti and I.E. Ragab, "A new generalized Lindley distribution:, *J. Stat. Comput. Simul.*, vol. 85, no. 18, pp. 3662-3678, 2015.

[49] Bhati, M.A. Malik and H.J. Vaman, "Lindley-exponential distribution: Properties and applications", *Metron*, vol. 73, no. 3, pp. 335-357, 2015.

[50] P.E. Oguntunde, A.O. Adejumo, H.I. Okagbue and M.K. Rastogi, "Statistical properties and applications of a new Lindley exponential distribution", *Gazi Uni. J. Sci.*, vol. 29, no. 4, pp. 831-838, 2016.

[51] S. Nedjar and H. Zeghdoudi, "On gamma Lindley distribution: Properties and Simulation *J. Comput. Appl. Math.*, vol. 298, pp. 167-174, 2016.

[52] W. Gui, S. Zhang and X. Lu, "The Lindley-Poisson distribution in lifetime analysis and its properties", *Hacttepe J. Math. Stat.*, vol. 43, no. 6, pp. 1063-1077, 2014.

[53] M.E. Ghitany, D.K. Al-Mutairi, F.A. Al-Awadhi and M.M. Al-Burais, "Marshall-Olkin extended Lindley distribution and its application", *Int. J. Appl. Math.*, vol. 25, no. 5, pp. 709-721, 2012.

[54] B.O. Oluyede and T. Yang, "A new class of generalized Lindley distributions with applications", *J. Stat. Comput. Simul.*, vol. 85, no. 10, pp. 2072-2100, 2015.

[55] M.M.E. Abd El-Monsef, "A new Lindley distribution with location parameter", *Comm. Stat. Theo. Meth.*, vol. 45, no. 17, pp. 5204-5219, 2016.

[56] V.K. Sharma, S.K. Singh, U. Singh and F. Merovci, "The generalized inverse Lindley distribution: A new inverse Statistical model for the study of upside-down bathtub data" *Comm. Stat. Theo. Meth.*, vol. 45, no. 19, pp. 5709-5729, 2015.

[57] S.H. Alkarni, "Extended inverse Lindley distribution: properties and application", *SpringerPlus*, vol. 4, no. 1, Article number 690, 2015.

[58] W.A. Hassaneim and T.A. Elhaddad, "Truncated Lindley Gamma distribution", *Pak. J. Stat*, vol. 32, no. 3, pp. 227-246, 2016.
[59] T. Hussain, M. Aslam and M. Ahmad, "A two parameter discrete Lindley distribution", *Rev. Colomb. Esta.*, vol. 39, no. 1, pp. 45-61, 2016.
[60] A. Al-Babtain, A.M. Eid, A. Ahmed and F. Merovci, "The five parameter Lindley distribution", *Pak. J. Stat.*, vol. 31, no. 4, pp. 363-384, 2015.
[61] W. Gui, "Statistical properties and applications of the Lindley slash distribution", *J. Appl. Stat. Sci.*, vol. 20, no. 3, pp. 283-298, 2012.
[62] S. Joshi and K.K. Jose, "Wrapped Lindley Distribution", *Comm. Stat. Theo. Meth.*, vol. 47, no. 5, pp. 1013-1021, 2018.
[63] S.A. Kemaloglu and M. Yilmaz, "Transmuted two-parameter Lindley distribution" *Comm. Stat. Theo. Meth.*, vol. 46, no. 23, pp. 11866-11879, 2017.
[64] M. Mesfioui and A.M. Abouammoh, "On a multivariate Lindley distribution", *Comm. Stat. Theo. Meth.*, vol. 46, no. 16, pp. 8027-8045, 2017.
[65] J. Mazucheli and J.A. Achcar, "The Lindley distribution applied to competing risks lifetime data", *Comp. Meth. Prog. Biomed.*, vol. 104, no. 2, pp. 188-192, 2011.
[66] A. A. Opanuga, H. I. Okagbue, E. A. Owoloko and O. O. Agboola, Modified Adomian Decomposition Method for Thirteenth Order Boundary Value Problems, *Gazi Uni. J. Sci*., vol. 30, no. 4, pp. 454-461, 2017.
[67] A.A. Opanuga, H.I. Okagbue and O.O. Agboola "Application of Semi-Analytical Technique for Solving Thirteenth Order Boundary Value Problem," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 145-148.
[68] A.A. Opanuga, H.I. Okagbue, O.O. Agboola, "Irreversibility Analysis of a Radiative MHD Poiseuille Flow through Porous Medium with Slip Condition," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp. 167-171.
[69] A.A. Opanuga, E.A. Owoloko, H.I. Okagbue, "Comparison Homotopy Perturbation and Adomian Decomposition Techniques for Parabolic Equations," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp. 24-27.
[70] A.A. Opanuga, E.A. Owoloko, H. I. Okagbue, O.O. Agboola, "Finite Difference Method and Laplace Transform for Boundary Value Problems," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp. 65-69.
[71] A.A. Opanuga, E.A. Owoloko, O.O. Agboola, H.I. Okagbue, "Application of Homotopy Perturbation and Modified Adomian Decomposition Methods for Higher Order Boundary Value Problems," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp. 130-134.
[72] A.A. Opanuga, S.O. Edeki, H.I. Okagbue, G.O. Akinlabi, A.S. Osheku and B. Ajayi, "On numerical solutions of systems of ordinary differential equations by numerical-analytical method", *Appl. Math. Sciences*, vol. 8, no. 164, pp. 8199 – 8207, 2014.
[73] S.O. Edeki , A.A. Opanuga, H.I. Okagbue , G.O. Akinlabi, S.A. Adeosun and A.S. Osheku, "A Numerical-computational technique for solving transformed Cauchy-Euler equidimensional equations of homogenous type. *Adv. Studies Theo. Physics*, vol. 9, no. 2, pp. 85 – 92, 2015.
[74] S.O. Edeki , E.A. Owoloko , A.S. Osheku , A.A. Opanuga , H.I. Okagbue and G.O. Akinlabi, "Numerical solutions of nonlinear biochemical model using a hybrid numerical-analytical technique", *Int. J. Math. Analysis*, vol. 9, no. 8, pp. 403-416, 2015.
[75] A.A. Opanuga , S.O. Edeki , H.I. Okagbue and G.O. Akinlabi, "Numerical solution of two-point boundary value problems via differential transform method", *Global J. Pure Appl. Math.*, vol. 11, no. 2, pp. 801-806, 2015.
[76] A.A. Opanuga, S.O. Edeki, H.I. Okagbue and G. O. Akinlabi, "A novel approach for solving quadratic Riccati differential equations", *Int. J. Appl. Engine. Res.*, vol. 10, no. 11, pp. 29121-29126, 2015.
[77] A.A Opanuga, O.O. Agboola and H.I. Okagbue, "Approximate solution of multipoint boundary value problems", *J. Engine. Appl. Sci.*, vol. 10, no. 4, pp. 85-89, 2015.
[78] A.A. Opanuga, O.O. Agboola, H.I. Okagbue and J.G. Oghonyon, "Solution of differential equations by three semi-analytical techniques", *Int. J. Appl. Engine. Res.*, vol. 10, no. 18, pp. 39168-39174, 2015.